The Microwave Background

The Big Bang model predicts the the universe was once significantly different than it is today. Specifically, the universe was hotter and denser. Let's consider some of the implications this has on observables.

The most direct evidence of the Big Bang is the cosmic microwave background radiation. Observations from the ground and from space put the temperature of the background at 2.73 K. Where precisely did this come from? Because of redshifting, the temperature of the microwave background has changed with redshift; specifically, since $\nu_e = \nu_0(1 + z)$,

$$T = T_0(1 + z) \qquad (22.01)$$

Consequently, the universe used to be substantially hotter than it is today. At one point in the past, the universe was so hot that all the hydrogen was ionized. During this time, the electrons were free to scatter any photon they encountered. This kept the photon mean free path short, and ensured that the energies of the photons were redistributed into a perfect blackbody. Eventually, however, the temperature dropped enough so that the electrons recombined to form (mostly) hydrogen atoms. At this time of "last scattering" the electrons could no longer interact with the photons, and the light was released.

What was the redshift of last scattering? Or, equivalently, when did most of the hydrogen atoms in the universe become neutral? Let's assume the universe is made entirely of hydrogen. Now, consider the Saha equation, which gives the fraction of an element in ionization state $i+1$ compared to that in ionization state i:

$$\frac{N_{i+1}}{N_i} N_e = 2 \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{u_{i+1}}{u_i} e^{-\chi/kT} \tag{22.02}$$

where u is the partition function (2 for neutral hydrogen, 1 for ionized hydrogen), χ the ionization energy of hydrogen (13.6 eV), and N the local density (in atoms per cubic centimeter). Obviously, for a pure hydrogen universe $N_e = N_{i+1}$ and $N = N_{i+1} + N_i$. If we let $x = N_{i+1}/N_i$ be the ionization fraction, then (3.12) reduces to

$$\frac{Nx^2}{1-x} = \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi/kT} \tag{22.03}$$

If we now substitute for N using $\rho = N m_H$, $T = T_0(1+z)$, and note that

$$\rho = \rho_0 \left(\frac{R_0}{R} \right)^3 = \rho_0 (1+z)^3 \tag{22.04}$$

then

$$\frac{x^2}{1-x} = \left(\frac{2\pi m_e kT_0}{h^2} \right)^{3/2} \frac{m_H}{\rho_0} (1+z)^{-\frac{3}{2}} e^{-\chi/kT_0(1+z)} \tag{22.05}$$

Recombination should occur when $x \approx 0.5$. With this substitution, the redshift of recombination becomes

$$(1+z) = \left(\frac{2\pi m_e kT}{h^2}\right)\left(\frac{m_H}{\rho_0}\right)^{2/3}\left(\frac{1-x}{x^2}\right)^{2/3} e^{-2\chi/3kT_0(1+z)} \tag{22.06}$$

(Note that you have to solve this numerically, since you have a $(1+z)$ in the exponent, as well as on the left hand side of the equal sign.) When solved, you find that recombination occurred at $z \sim 1500$, at a recombination temperature of $T_{rec} \sim 4000°K$. For an Einstein-de Sitter $H_0 = 70$ km/s/Mpc universe, this corresponds to an age of 170,000 yr; if $\Lambda = 0.7$, $\Omega_M = 0.3$, then $t = 215,000$ yr.

The Radiation Dominated Era

If you go back further than the surface of last scattering, you reach a time where the radiation density was greater than the matter density. At that time, the equation of motion for the universe was slightly different. Unlike matter, whose density declines as $1/R^3$, radiation density declines as $1/R^4$ (due to the redshifting of the photon's energy). Thus the Friedmann equation is

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{R_0}{R}\right)^4 = -\frac{kc^2}{R^2} \qquad (22.07)$$

Now, in the early universe, R is small, so $G\mathcal{M}/R \gg E$. Given that the potential (and kinetic) energy terms completely dwarf the total energy term, you can make the approximation $E \approx 0$. Thus, you are left with the differential equation

$$\dot{R}^2 - \frac{8}{3}\pi G\rho_0 R_0^4 R^{-2} = 0 \qquad (22.08)$$

(The exponent on the second R is -2, instead of -1 in the matter dominated universe.) Thus, the solution to the equation is slightly different,

$$R = bt^{1/2} \qquad (22.09)$$

with

$$b = \left\{\frac{32}{3}\pi G\rho_0\right\}^{1/4} R_0 \qquad (22.10)$$

So the expansion of the universe proceeded at a different rate in the era of radiation.

The Creation of Helium

The Big Bang model also allows us to predict the helium content of the universe. As equation (22.01) states, the higher the redshift, the higher the temperature of the universe. Now, consider a time when the universe was extremely hot, with a temperature $T_N > 0.511$ MeV (or, in terms of Kelvins, about 10 billion degrees). At these temperatures, a photon passing a charged particle can spontaneously be converted into an electron-positron pair (this is called pair-production). So early-on, there were a large number of reactions which turned protons into neutrons (and vice-versa) via

$$\begin{aligned} p + e^- &\rightleftharpoons n + \nu \\ p + \bar{\nu} &\rightleftharpoons n + e^+ \end{aligned} \tag{22.11}$$

Under equilbrium conditions, one can consider the neutron to be just an "excited state" of a proton. In other words, the ratio of neutrons to protons can be expressed via the Boltzmann equation

$$\frac{N_n}{N_p} = \exp\left\{-\frac{(m_n - m_p)c^2}{kT}\right\} \tag{22.12}$$

where $(m_n - m_p)c^2$ is the energy difference between the state of being a neutron and the state of being a proton. In this early primordial soup, neutrons have a very small mean free path, so that, once formed, virtually every neutron immediately interacts with a proton to create deuterium (and then, eventually, helium). Let's assume that all neutrons end up in helium atoms. Since a helium atom consists of 2 protons and 2 neutrons, the number of helium atoms will be $N_{\rm He} = N_n/2$. Moreover, since the total density of particles to start with is $N_n + N_p$, the fractional abundance (by mass) of helium will be

$$Y = \frac{4(N_n/2)}{N_n + N_p} = \frac{2(N_n/N_p)}{1 + (N_n/N_p)} \tag{22.13}$$

But, note that through (22.12), N_n/N_p is a function of temperature. So the mass fraction of helium in the universe is constantly changing, and

$$Y = \frac{2\exp\left\{-\dfrac{(m_n - m_p)c^2}{kT}\right\}}{1 + \exp\left\{-\dfrac{(m_n - m_p)c^2}{kT}\right\}} = \frac{2}{1 + \exp\left\{+\dfrac{(m_n - m_p)c^2}{kT}\right\}} \tag{22.14}$$

This relation will hold until the temperature of the universe drops to a number sufficiently low so that pair-production (*i.e.,* neutron production) is "frozen out." When does this occur? Well, once the temperature drops below 0.511 MeV, there is no longer enough energy for pair production, and neutron creation will stop. However, in practice, the freeze-out occurs slightly earlier than this, since the reactions given in (22.11) do not happen instantaneously. In other words, statistical equilbrium between protons and neutrons will stop when the timescale for neutron creation becomes longer than the time until $T_N = 0.511$ MeV. This depends on the expansion rate of the universe (and therefore the matter density of the universe). If neutrons are frozen out at a temperature of ~ 1 MeV, the helium fraction of the universe will be $Y = 0.43$; if freezing out occurs at $T_N \sim 0.7$ MeV, then $Y = 0.27$, and if $T_N \sim 0.511$ MeV, $Y = 0.14$. The observed helium fraction of old Pop II stars is $Y \sim 0.23$, which translates to $T_N \sim 0.64$ MeV.

Growth of Structure

At the time of the Big Bang, the universe was extremely smooth, but today it is extremely clumpy. How did we get to this state?

First, let's define two classes of perturbations. *Isothermal perturbations* are fluctuations that affect the matter of the universe, but not the surrounding radiation field. These are the types of fluctuations that occur today, since matter and radiation are decoupled. But in the early universe (in the era before recombination), this type of fluctuation may not have existed.

The second class of perturbations are called *adiabatic perturbations*. These effect both light and matter together. Such perturbations do not occur today, but they might have occurred at early times before decoupling.

Isothermal Perturbations

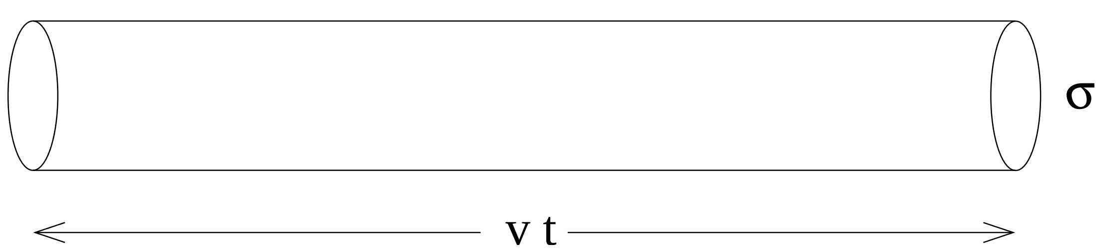
Fluctuations grow due to gravity; the gravitational force per unit mass is

$$F_{\text{grav}} = \frac{G\mathcal{M}}{R^2} = \frac{4}{3}\pi G\rho R \tag{22.15}$$

Now consider an isothermal perturbation, where gravity is attempting to move electrons (and protons) through a radition field. The velocity of the electrons is v; the cross section of the electrons to photon absorption is σ_T (the Thomson cross section). Because the electrons are moving through a medium filled with photons, they will experience a drag force. Since the protons must follow the electrons, they will also (indirectly) feel this force. Per unit mass, the drag force is

$$F_{\text{drag}} = \frac{aT^4(\sigma_T\, v)}{m_H c} \tag{22.16}$$

(It's not difficult to see where this equation comes from: aT^4 is the energy density (ergs/cm^3) of the microwave background, and $\sigma_T\, v$ is the cylindrical volume swept out by the electron each second. So the numerator is the amount of energy encountered by the electron each second as it plows through the photons.)



If we compare F_{drag} to F_{grav}, and explicitly put in the redshift dependence of T (22.01) and ρ (22.04) then

$$\frac{F_{\text{drag}}}{F_{\text{grav}}} = \frac{3\sigma_T\, vaT^4}{4\pi G m_H c R\,\rho} = \frac{3\sigma_T\, v\, aT_0^4(1+z)^4}{4\pi G m_H c R\,\rho_0(1+z)^3} \tag{22.17}$$

Now consider R, the physical size of the perturbation being considered. In an Einstein de-Sitter universe, the time it takes an electron to move a distance R is approximately $v\,t$. If perturbations begin at the time of Big Bang, then by redshift z, the size is

$$R = \frac{2}{3}\frac{1}{H_0}v(1+z)^{-3/2} \tag{22.18}$$

If we use this to substitute for R, then

$$\frac{F_{\rm drag}}{F_{\rm grav}} = \frac{9\,\sigma_T a T_0^4 H_0}{8\pi G m_H c\,\rho_c\,\Omega_0}(1+z)^{5/2} \tag{22.19}$$

Numerically, this works out to be

$$\frac{F_{\rm drag}}{F_{\rm grav}} \sim 10^{-8}(1+z)^{5/2} \tag{22.20}$$

So note: before decoupling, when $z > 1500$, gravity cannot overcome the drag force of the microwave photons. So isothermal perturbations during this era do not grow.

Adiabatic Perturbations

If the radiation field is perturbed along with the matter, the excess energy contained in a region will diffuse outward and damp out the fluctuation. However, this damping does not occur instantaneously.

Let $\lambda = 1/(n_e \sigma_T)$ be the mean free path of a photon. In terms of λ, the time between scatterings for photons is λ/c, and, from statistics, the number of scatterings necessary for a photon to random walk a distance R is $(R/\lambda)^2$. For a fluctuation *not* to be damped out, the diffusion time must be longer than the age of the universe, *i.e.*,

$$\left(\frac{R}{\lambda}\right)^2 \left(\frac{\lambda}{c}\right) > \frac{2}{3}\frac{1}{H_0}(1+z)^{-3/2} \quad \Longrightarrow \quad R^2 > \frac{2}{3}\frac{c\,\lambda}{H_0}(1+z)^{-3/2} \tag{22.21}$$

Now if we substitute mass for radius and again note that the protons must follow the electrons, so that $\rho = n_e m_H$, then the condition for perturbation growth is

$$\mathcal{M} = \frac{4}{3}\pi\rho R^3 > \frac{4}{3}\pi\rho \left\{ \frac{2c\,m_H}{3H_0\rho\,\sigma_T}(1+z)^{-3/2} \right\}^{3/2} \tag{22.22}$$

or, if we use (22.04) to substitute for density,

$$\mathcal{M} > \frac{4\pi}{3}\left\{\frac{2cm_H}{3H_0\,\sigma_T}\right\}^{-3/2}\left\{\frac{1}{\rho_c\Omega_0}\right\}^{1/2}(1+z)^{-15/4} \tag{22.23}$$

where ρ_c is the critical density of the universe. At decoupling, this works out to $\mathcal{M} \gtrsim 10^{13}\mathcal{M}_\odot$. This is somewhat of an underestimate, since λ is changing rapidly in the early universe. But it does demonstrate that adiabatic perturbations can only propagate if they are very large; small scale perturbations will be damped out.

The Jeans Mass

Another question we can ask concerns the minimum mass for gravitational collapse. For collapse to occur, the gravitational potential energy must overcome the thermal motion of the gas. If v_s is the sound speed,

$$\frac{G\mathcal{M}}{R} > \frac{1}{2} v_s^2 \tag{22.24}$$

If we substitute density for radius, then

$$2\, G\mathcal{M} \left(\frac{4\pi\rho}{3\mathcal{M}} \right)^{1/3} > v_s^2 \tag{22.25}$$

so

$$\mathcal{M}_J = \frac{1}{4} \left(\frac{3}{2\pi} \right)^{\frac{1}{2}} G^{-\frac{3}{2}} \rho^{-\frac{1}{2}}\, v_s^3 \tag{22.26}$$

where $\mathcal{M}_J$ is called the Jeans mass. After de-coupling, matter will act as an ideal gas, so

$$v_s = \left(\frac{\gamma P}{\rho} \right)^{1/2} = \left(\frac{\gamma k T}{m_H} \right)^{1/2} \tag{22.27}$$

If we plug in the numbers for the universe shortly after decoupling *i.e.*, $z \sim 1000$, then $v_s \sim 4$ km s^{-1}, and the Jeans mass is $\mathcal{M}_J \sim 10^5 \mathcal{M}_\odot$. This is similar to the mass of a globular cluster.

In the early universe, however, the ideal gas law did not apply. At $z \sim 40,000$, radiation pressure dominated, and the energy density was much greater than the matter density. So

$$P = \frac{aT^4}{3} \quad \text{and} \quad \rho = \frac{aT^4}{c^2} \tag{22.28}$$

During this time, the sound speed was

$$v_s \sim \frac{c}{\sqrt{3}} \tag{22.29}$$

and the density was

$$\rho = \frac{a}{c^2} T_0^4 (1+z)^4 \tag{22.30}$$

The Jeans mass was therefore $\mathcal{M}_J \sim 10^{16} \mathcal{M}_\odot$. Again this demonstrates that large fluctuations could grow in the early universe.

Growth of Perturbations

Can we connect the structure we see today to perturbations in the microwave background? The largest structures in today's universe are superclusters; these have overdensities of $\delta\rho/\rho \sim 2$. To see what that means, consider a region of slightly enhanced density, $\delta\rho$ within an Einstein de-Sitter universe. Recall from our discussion of the basic equations of cosmology (2.02), the expansion of any region of space is given by

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G(\rho + \delta\rho) = -\frac{kc^2}{R^2} \tag{22.31}$$

which, for $\rho > \rho_c$, has the solution

$$R = a(1 - \cos\theta) \qquad t = \frac{a}{c}(\theta - \sin\theta) \tag{22.32}$$

Let's just consider the R part of the solution. In the early universe, θ was small, so we can Taylor expand the trignometric terms as

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \ldots \tag{22.33}$$

So (22.32) becomes

$$R = a\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 + \ldots\right) \tag{22.34}$$

If we assume the perturbation is small and neglect the higher order terms, then

$$R + \delta R = a\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right) \tag{22.35}$$

so

$$R \propto \theta^2 \quad \text{and} \quad \delta R \propto \theta^4 \implies \frac{\delta R}{R} \propto \theta^2 \propto R \tag{22.36}$$

Now let's relate this to the density.

$$\rho = \frac{\mathcal{M}}{R^3} \implies \delta\rho = -\frac{3\mathcal{M}}{R^4}\delta R = -\frac{3\rho}{R}\delta R \tag{22.37}$$

so

$$\frac{\delta\rho}{\rho} = 3\frac{\delta R}{R} \tag{22.38}$$

or

$$\frac{\delta\rho}{\rho} \propto \frac{\delta R}{R} \propto R \propto (1+z)^{-1} \tag{22.39}$$

In other words, the amplitude of a small density perturbation will grow linearly with the size of the universe due to the Hubble expansion alone, *i.e.,* without any assistance from gravity. This puts a limit on the amplitude of the initial density perturbations: for example, if a supercluster now has $\delta\rho/\rho \sim 2$, at decoupling, the initial density fluctuation must have been at least 1000 times smaller.

Note, however that very early on, when the universe was dominated by radiation, $\rho \propto R^{-4}$ (since the dominate source of matter/energy was being redshifted). So at this time

$$\frac{\delta\rho}{\rho} \propto R^2 \tag{22.40}$$

One can then relate these to t through (1.29) and (22.09). In the matter dominated era,

$$\frac{\delta\rho}{\rho} \propto R \propto t^{2/3} \tag{22.41}$$

while at earlier times

$$\frac{\delta\rho}{\rho} \propto R^2 \propto t \tag{22.42}$$

The Sachs-Wolfe Relation

We can observe the density fluctuations of the universe directly in the microwave background via the Sachs-Wolfe Relation. Consider a density contrast $\delta_x = \delta\rho/\rho$ over some region of the universe, which, in co-moving coordinates is u. Because the universe is expanding, the physical size of this region will expand with time, so that the actual size of the region is Ru. Now consider the gravitational potential of the region. From simple Newtonian physics, this is

$$\phi = \frac{G\mathcal{M}}{Ru} = \frac{4\pi G R^3 u^3 \delta_x \rho}{3Ru} = \frac{4}{3}\pi G R^2 u^2 \delta_x \rho \qquad (22.43)$$

Next, let's substitute for the mean density of the universe. If we use the definition of Ω (1.18), then

$$\rho = \Omega \rho_c = \frac{3}{8\pi G} H^2 \Omega \qquad (22.44)$$

and

$$\phi = \frac{1}{2}\Omega\, R^2 u^2 H^2 \delta_x \qquad (22.45)$$

Let's examine this equation. If we make the (good) approximation that the density of the universe at high-z was very near critical (*i.e.,* the Einstein-de Sitter equations hold), then from (1.29) and (1.32)

$$R \propto t^{2/3} \quad \text{and} \quad H \propto \frac{1}{t} \quad \Longrightarrow \quad H \propto R^{-3/2} \qquad (22.46)$$

Since from (22.42), $\delta_x \propto R$

$$\phi \propto \Omega R^2 R^{-3} R \quad \Longrightarrow \quad \phi = \text{Constant} \qquad (22.47)$$

In other words, although the fluctuation grows with time, the potential due to the fluctuation does not. We can therefore substitute into (22.45) the current values of the universe

$$\phi = \frac{1}{2}\Omega_0 H_0^2 (R_0 u)^2 \delta_x(0) \tag{22.48}$$

Now a photon passing through this potential will experience a gravitational redshift of the order of

$$\frac{\delta\lambda}{\lambda} = \frac{\phi}{c^2} \tag{22.49}$$

Thus, the microwave photons of the region will appear cooler by

$$\frac{\delta T}{T} = \frac{\Omega_0 H_0^2 (R_0 u)^2 \delta_x(0)}{2c^2} \tag{22.50}$$

or

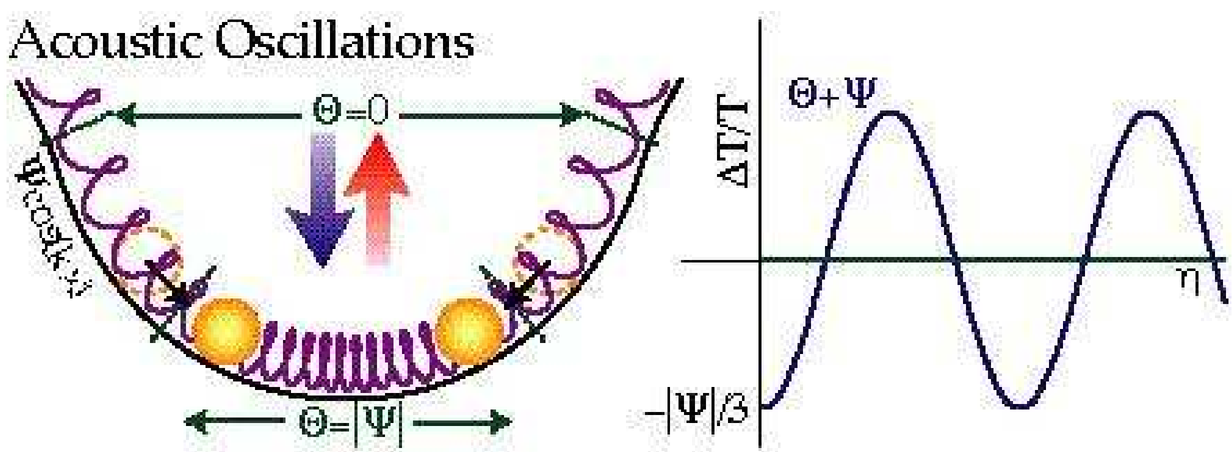
$$\frac{\delta T}{T} \propto \frac{\delta\rho}{\rho} \tag{22.51}$$

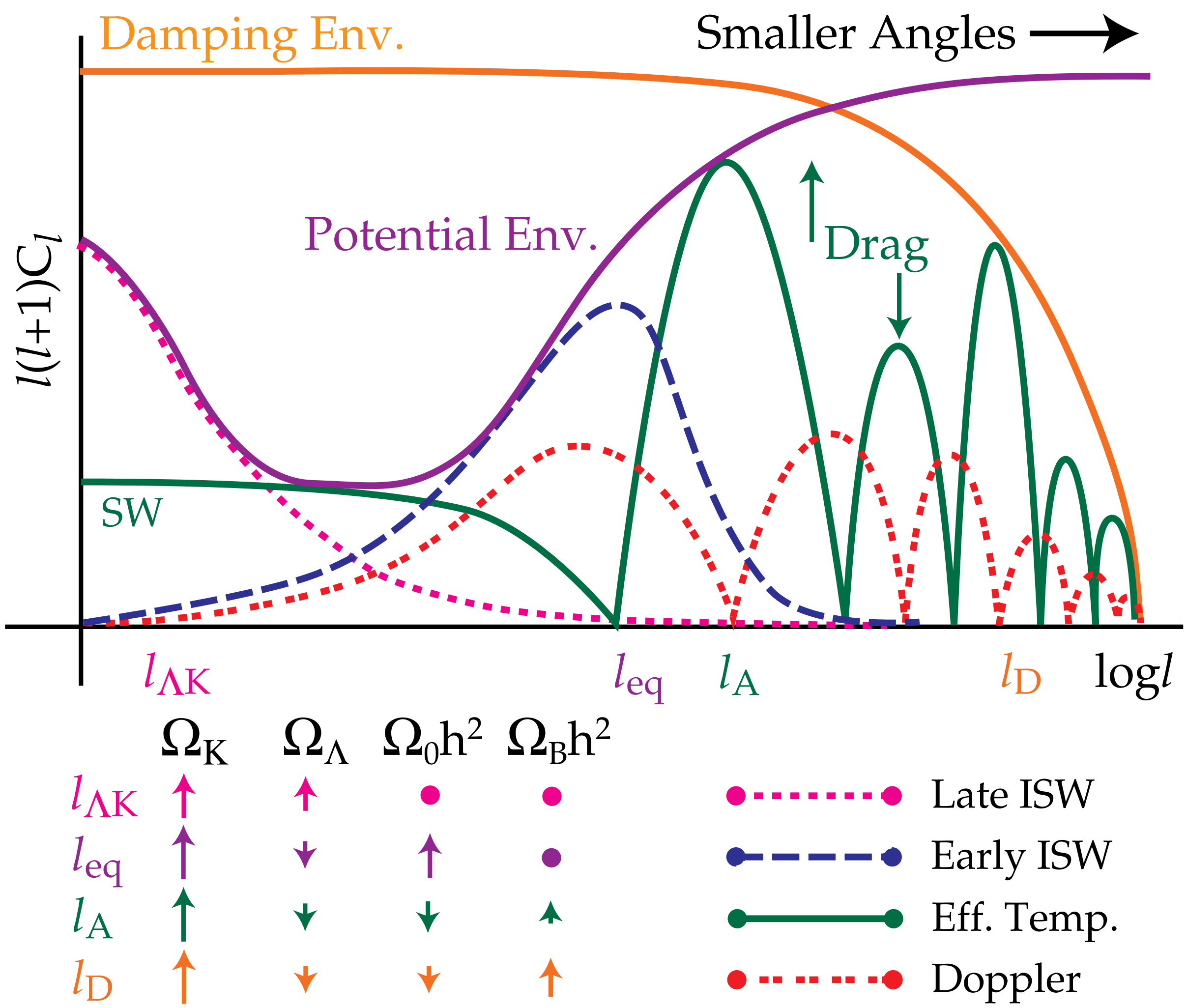
Let's put in some typical numbers. Superclusters have typical sizes of $Ru \sim 100$ Mpc and density contrasts of $\delta_x \sim 2$. For a Hubble Constant of $H_0 = 75$ km s^{-1} Mpc^{-1}, this means that $\delta T/T \sim 6 \times 10^{-4}$ Mpc. This is not too far from the $\sim 3 \times 10^{-5}$ value that is observed. (Note that this effect is most easily observable at large scales: at small scales, the gravitational redshift is much harder to detect (and can be lost in the noise of other sources.)

The above argument is valid if the potential well doesn't change while the photon crosses it. But if the well does change, differential effects can induce a further anistropy. Simply put, the blueshift acquired as the photon falls into the potential isn't the same as the redshift it feels as it climbs out. (There's also an additional GR effect, associated with the stretching of space-time by the potential.) This is called the Integrated Sachs-Wolfe Effet.

Acoustical Peaks and Microwave Anisotropy

As we have seen, small scale oscillations are subject to damping by radiation pressure. The physics of this damping is similar to that of a simple harmonic oscillator, with the potential well forcing matter in, and radiation pressure driving it out. The shorter the wavelength of the potential fluctuation, the faster the fluid oscillates, such that at the last scattering, the phase of the oscillation reaches the same scale as the wavelength. Regions of compression (maxima) will be hot regions; areas of rarefaction (minima) cold regions. Consequently, there will be a harmonic series of peaks in the wavelength associated with oscillations.




Damping Env.
Smaller Angles
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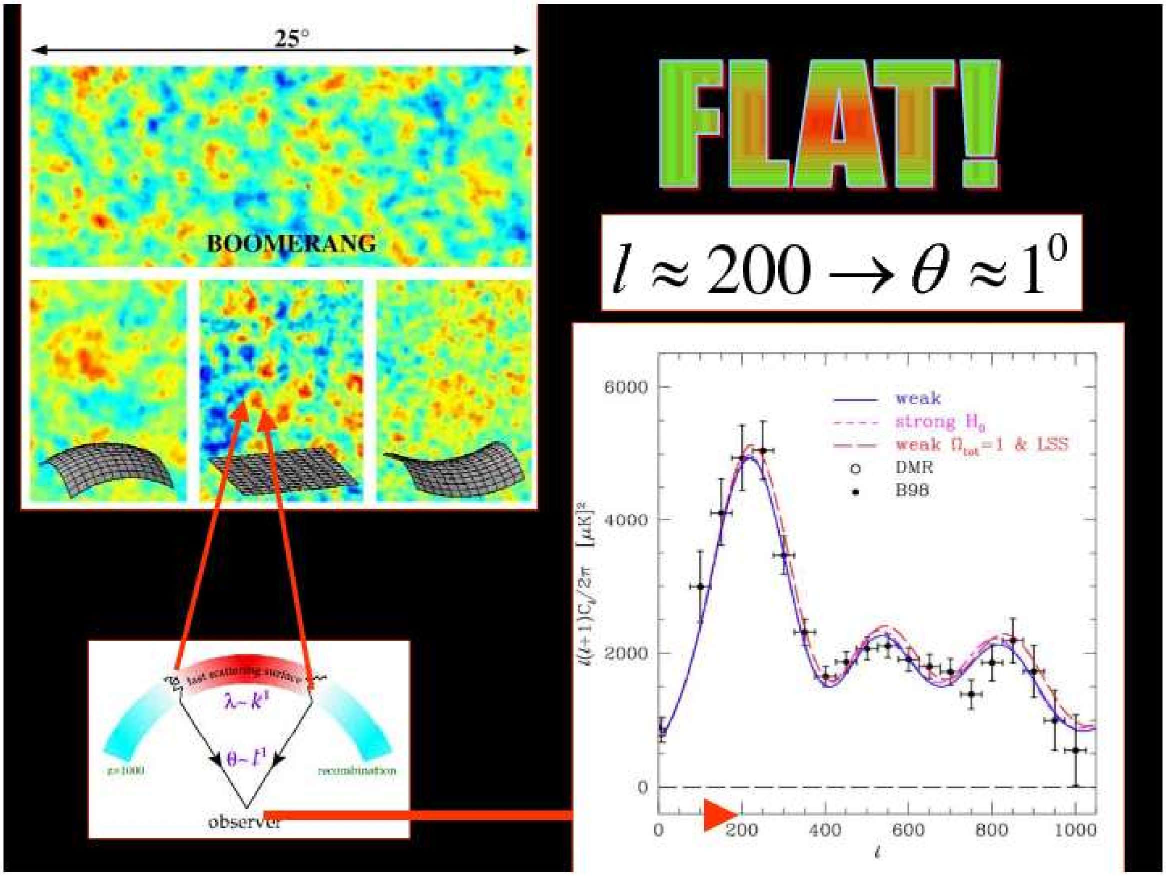
25°
BOOMERANG
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$l \approx 200 \rightarrow \theta \approx 1^0$
observer
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